Transversals of Longest Paths and Cycles

Dieter Rautenbach[1] and Jean-Sébastien Sereni[2]

[1] Institut für Optimierung und Operations Research
Universität Ulm, Ulm, Germany, dieter.rautenbach@uni-ulm.de

[2] Centre national de la recherche scientifique
LORIA, Vandœuvre-lès-Nancy, France, sereni@kam.mff.cuni.cz

Abstract

Let G be a graph of order n. Let $\mathrm{lpt}(G)$ be the minimum cardinality of a set X of vertices of G such that X intersects every longest path of G and define $\mathrm{lct}(G)$ analogously for cycles instead of paths. We prove that

- $\mathrm{lpt}(G) \leqslant \left\lceil \frac{n}{4} - \frac{n^{2/3}}{90} \right\rceil$, if G is connected,
- $\mathrm{lct}(G) \leqslant \left\lceil \frac{n}{3} - \frac{n^{2/3}}{36} \right\rceil$, if G is 2-connected, and
- $\mathrm{lpt}(G) \leqslant 3$, if G is a connected circular arc graph.

Our bound on $\mathrm{lct}(G)$ improves an earlier result of Thomassen and our bound for circular arc graphs relates to an earlier statement of Balister *et al.* the argument of which contains a gap. Furthermore, we prove upper bounds on $\mathrm{lpt}(G)$ for planar graphs and graphs of bounded tree-width.

Keywords: Longest path, longest cycle, transversal.

MSC2010: 05C38, 05C70

1 Introduction

It is well known that every two longest paths in a connected graph as well as every two longest cycles in a 2-connected graph intersect. While these observations are easy exercises, it is an open problem, originating from a question posed by Gallai [2], to determine the largest value of k such that for every connected graph and every k longest paths in that graph, there is a vertex that belongs to all of these k paths. The above remark along with examples constructed by Skupień [5] ensure that $2 \leqslant k \leqslant 6$.

We consider only simple, finite, and undirected graphs and use standard terminology. For a graph G, we define $\mathcal{P}(G)$ to be the collection of all longest paths of G and a *longest path transversal* of G to be a set of vertices that intersects every longest path of G. Let $\mathrm{lpt}(G)$ be the minimum cardinality of a longest path transversal of G. We define $\mathcal{C}(G)$, a *longest cycle transversal*, and the parameter $\mathrm{lct}(G)$ analogously for cycles instead of paths.

The intersections of longest paths and cycles have been studied in detail and Zamfirescu [8] gave a short survey. In the present paper we prove upper bounds on $\mathrm{lpt}(G)$ and $\mathrm{lct}(G)$. Our bound on $\mathrm{lct}(G)$ for a 2-connected graph G improves an earlier result of Thomassen [6]. Balister *et al.* [1] showed that for every connected interval graph, there is a vertex that belongs to every longest path. Furthermore, their work [1] contains the statement that for every connected circular arc graph, there is a vertex that belongs to every longest path. Unfortunately, we believe that the argument they provide has a gap. We shall explain the approach of Balister *et al.*, the problem with their argument, and give a proof of a weaker result, specifically that every connected circular arc graph contains a longest path transversal of order at most 3.

2 Results

We start by proving a lemma that allows us to exploit the structure of some particular matchings to find long paths and cycles.

Lemma 1. *If $G = (P \cup Q) + M$ where $P : u_1 \dots u_\tau$ and $Q : v_1 \dots v_\tau$ are paths and M is a matching of edges between $V(P)$ and $V(Q)$ that has a partition $M = M_1 \cup \dots \cup M_q$ such that*

(a) $|M_i|$ is either 1 or even for $i \in [q]$ and

(b) if $u_{i_1}v_{i_2} \in M_i$ and $u_{j_1}v_{j_2} \in M_j$ for $i, j \in [q]$, then

$$(j_1 - i_1)(j_2 - i_2) \begin{cases} < 0, & \text{if } i = j \text{ and} \\ > 0, & \text{if } i \neq j, \end{cases}$$

that is, the edges in one of the sets M_i are pairwise "crossing" *and the edges in distinct sets M_i are pairwise* "parallel",

then G contains a path between a vertex in $\{u_1, v_1\}$ and a vertex in $\{u_\tau, v_\tau\}$ of order at least $\tau + |M|$.

Proof. If $i_0 = 1$, $i_{|M|+1} = \tau$, and $u_{i_1}, \dots, u_{i_{|M|}}$ with $1 \leqslant i_1 < \dots < i_{|M|} \leqslant \tau$ are the vertices of P that are incident with edges in M, then a subpath of P of the form $u_{i_j} \dots u_{i_{j+1}}$ with odd/even j is called an *odd/even segment of P*, respectively. Odd/even segments of Q are defined analogously.

The odd segments of P, M, and the even segments of Q define a path P'. Similarly, the even segments of P, M, and the odd segments of Q define a path Q'. See Figure 1 for an illustration. Since $E(P') \cap E(Q') = M$ and $E(P') \cup E(Q') = E(P) \cup E(Q) \cup M$, the longer of the two paths satisfies the desired properties. $\square$

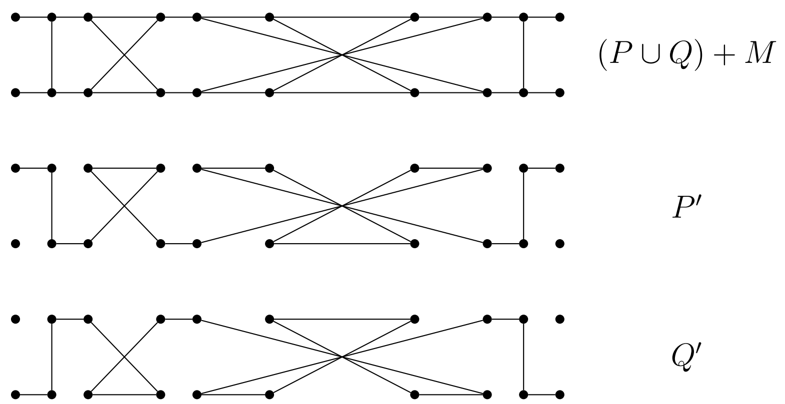

Figure 1: The two paths P' and Q' for a set M with $|M_1| = |M_4| = 1$, $|M_2| = 2$, and $|M_3| = 4$.

We proceed to our first main result. Note that in the proof of Theorem 2, as well as of Theorem 3 below, we did not try to minimize the factor of $n^{\frac{2}{3}}$. The point of these two results is that $\mathrm{lpt}(G)$ is strictly less than $n/4$ and $\mathrm{lct}(G)$ is strictly less than $n/3$, respectively.

Theorem 2. *If G is a connected graph of order n, then* $\mathrm{lpt}(G) \leqslant \left\lceil \frac{n}{4} - \frac{n^{2/3}}{90} \right\rceil$.

Proof. Let G be a connected graph of order n. Let $\epsilon = \frac{1}{90} n^{-\frac{1}{3}}$ and $\tau = \left\lceil \left(\frac{1}{4} - \epsilon \right) n \right\rceil$. For a contradiction, we assume that $\mathrm{lpt}(G) > \tau$. Let $P : u_1 \dots u_\ell$ be a longest path of G. Since $V(P)$ as well as every set of $n - \ell + 1$ vertices of G are longest path transversals, we obtain

$$\left(\frac{1}{4} - \epsilon \right) n \leqslant \tau < \ell < n - \tau + 1 \leqslant \left(\frac{3}{4} + \epsilon \right) n + 1. \tag{1}$$

Let $p = \left\lceil \frac{\ell - \tau}{2} \right\rceil$. Since the set $T = \{u_i : p + 1 \leqslant i \leqslant p + \tau\}$ is too small to be a longest path transversal of G, there is a path $P' : v_1 \dots v_\ell$ in $G - T$. Since G is connected, the paths P and P' intersect.

If $V(P) \cap V(P') \subseteq \{u_1, \dots, u_p\}$, then let $v_x = u_r \in V(P) \cap V(P')$ be such that r is maximum. We may assume that $x \geqslant \frac{\ell+1}{2}$. Now $v_1 \dots v_x u_{r+1} \dots u_\ell$ is a path of order at least $x + \ell - p \geqslant \frac{\ell+1}{2} + \ell - \frac{\ell - \tau + 1}{2} = \ell + \frac{\tau}{2} > \ell$, which is a contradiction. Hence P and P' intersect in a vertex in $\{u_1, \dots, u_p\}$ as well as a vertex in $\{u_{p+\tau+1}, \dots, u_\ell\}$. Let $v_x = u_r$ be in $V(P') \cap \{u_1, \dots, u_p\}$ such that r is maximum and $v_y = u_s$ be in $V(P') \cap \{u_{p+\tau+1}, \dots, u_\ell\}$ such that s is minimum. We may assume that $x < y$.

Since $v_1 \dots v_x u_{r+1} \dots u_{s-1} v_y \dots v_\ell$ is a path of order at least $\ell - (y - x - 1) + \tau$, we obtain $y - x - 1 \geqslant \tau$. Since $u_{s-1} \dots u_{r+1} v_x \dots v_\ell$ is a path of order at least $\tau + \ell - (x - 1)$, we obtain $x - 1 \geqslant \tau$. Since $u_{r+1} \dots u_{s-1} v_y \dots v_1$ is a path of order at least $\tau + y$, we obtain $\ell - y \geqslant \tau$.

Choosing four vertex-disjoint paths $A : a_1 \dots a_\tau$, $B : b_1 \dots b_\tau$, $C : c_1 \dots c_\tau$, and $D : d_1 \dots d_\tau$ as subpaths of the four paths $P'[\{v_1, \dots, v_{x-1}\}]$, $P[\{u_{r+1}, \dots, u_{s-1}\}]$, $P'[\{v_{x+1}, \dots, v_{y-1}\}]$, and $P'[\{v_{y+1}, \dots, v_\ell\}]$, respectively, we obtain the existence of two vertex-disjoint sets X and Y in

$V(G) \setminus (V(A) \cup V(B) \cup V(C) \cup V(D))$ with $|X \cup Y| \leqslant n - 4\tau \leqslant 4\epsilon n$ such that X contains a path between some neighbors of any two of the vertices a_1, b_1, and c_1, and Y contains a path between some neighbors of any two of the vertices b_τ, c_τ, and d_1. See Figure 2 for an illustration.

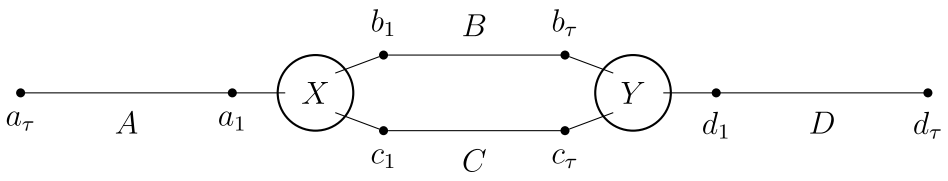

Figure 2: The paths A, B, C, and D and the two sets X and Y.

If $a_i b_j$ is an edge of G with $j \geqslant i$, then the path $a_\tau \dots a_i b_j \dots b_1$, a path in X between neighbors of b_1 and c_1, the path C, a path in Y between neighbors of c_τ and d_1, and the path D form a path of order at least $3\tau + (j - i + 1) + 2$. By (1), this implies that

$$j - i \leqslant \lceil 4\epsilon n \rceil - 3. \tag{2}$$

Our goal is now to prove the existence of a vertex cover $T_{A,B}$ of small order for the bipartite graph $G_{A,B}$ with bipartition $V(A)$ and $V(B)$ formed by the edges between these two sets. First, note that if $4\epsilon n \leqslant 2$, then (2) implies that this bipartite graph is edgeless, so it is enough to set $T_{A,B} = \emptyset$.
Assume now that $4\epsilon n > 2$. Let N be a maximum matching of $G_{A,B}$. Let $I = \left[\left\lceil \frac{\tau}{2(\lceil 4\epsilon n \rceil - 3)} \right\rceil\right]$. For $i \in I$, let N_i be the set of edges in N that are incident with a vertex in

$$\{a_j \,:\, 2(\lceil 4\epsilon n \rceil - 3)(i-1) + 1 \leqslant j \leqslant \min\{\tau, 2(\lceil 4\epsilon n \rceil - 3)i\}\}.$$

By (2), if $a_{i_1} b_{i_2} \in N_i$ and $a_{j_1} b_{j_2} \in N_j$ with $j - i \geqslant 2$, then $(j_1 - i_1)(j_2 - i_2) > 0$, that is, the two edges are parallel in the sense of Lemma 1. Without loss of generality, we may assume that $\bigcup_{i \in I \,:\, i \text{ odd}} N_i$ contains at least half the edges of N. Since N is a matching, $|N_i| \leqslant 2(\lceil 4\epsilon n \rceil - 3)$ for every $i \in I$. Since permutation graphs are perfect [3, Chapter 7], each N_i contains a set of at least $\sqrt{|N_i|} \geqslant \frac{|N_i|}{\sqrt{2(\lceil 4\epsilon n \rceil - 3)}}$ edges that are either all pairwise parallel or all pairwise crossing in the sense of Lemma 1. This implies that N contains a subset M_0 that satisfies condition (b) from Lemma 1 with $|M_0| \geqslant \sum_{i \in I \,:\, i \text{ odd}} \frac{|N_i|}{\sqrt{2(\lceil 4\epsilon n \rceil - 3)}} \geqslant \frac{1}{2\sqrt{2(\lceil 4\epsilon n \rceil - 3)}} |N|$. By removing a set of at most $|M_0|/3$ edges from M_0, we obtain a matching M that satisfies both conditions from Lemma 1 with $|M| \geqslant \frac{2}{3} |M_0| \geqslant \frac{1}{3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}} |N|$.

If N has order at least $(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$, then N contains a matching M as in Lemma 1 of order at least $4\epsilon n - 1$. Thus by Lemma 1, the graph $(A \cup B) + M$ contains a path Q between $\{a_1, b_1\}$ and $\{a_\tau, b_\tau\}$ of order at least $\tau + |M|$. Now the path Q, a path in X between neighbors of a vertex in $\{a_1, b_1\}$ and c_1, the path C, a path in Y between neighbors of c_τ and d_1, and the path D form a path of order at least $3\tau + |M| + 2 \geqslant \left(\frac{3}{4} + \epsilon\right) n + 1$, which contradicts (1). Hence the bipartite graph $G_{A,B}$ has no matching of order at least $(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$. Now König's theorem [4] implies that $G_{A,B}$ has a vertex cover $T_{A,B}$ of order less than $(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$.

Similar arguments yield that for every two distinct paths $Q, R \in \{A, B, C, D\}$, the bipartite graph $G_{Q,R}$ with bipartition $V(Q)$ and $V(R)$ formed by the edges between these two sets has a vertex cover $T_{Q,R}$ of order less than $(4\epsilon n-1)3\sqrt{2(\lceil 4\epsilon n\rceil - 3)}$ if $4\epsilon n > 2$ and of order 0 otherwise. (If, for instance, $a_i d_j$ is an edge of G with $j \geqslant i$, then the path $a_\tau \dots a_i d_j \dots d_1$, a path in Y between neighbors of d_1 and b_τ, the path B, a path in X between neighbors of b_1 and c_1, and the path C again form a path of order at least $3\tau + (j - i + 1) + 2$ and we can argue as above.)

Let $T' = X \cup Y \cup \bigcup_{\{Q,R\} \in \binom{\{A,B,C,D\}}{2}} T_{Q,R}$. Since every component of $G - T'$ has order at most τ, the set T' is a longest path transversal of G of order less than $4\epsilon n + 6(4\epsilon n-1)3\sqrt{2(\lceil 4\epsilon n\rceil - 3)}$ if $4\epsilon n > 2$ and of order at most $4\epsilon n$ otherwise. For $\epsilon = \frac{1}{90}n^{-\frac{1}{3}}$, it follows that $|T'| \leqslant \left(\frac{1}{4} - \epsilon\right) n$, which yields the final contradiction. □

For the fractional version of the longest path transversal problem, a much stronger result is possible. In fact, for every connected graph G, there is a function $t \colon V(G) \to [0, 1]$ such that

$$\sum_{u \in V(G)} t(u) \leqslant \sqrt{n} \quad \text{and}$$
$$\sum_{u \in V(P)} t(u) \geqslant 1 \quad \text{for every } P \in \mathcal{P}(G).$$

Indeed, if the largest order of the paths in $\mathcal{P}(G)$ is at most $\sqrt{n}$, then let t be the characteristic function of $V(P)$ for some $P \in \mathcal{P}(G)$, otherwise let t be the constant function of value $\frac{1}{\sqrt{n}}$.

Confirming a conjecture by Zamfirescu [7], Thomassen [6] proved that $\mathrm{lct}(G) \leqslant \left\lceil \frac{|V(G)|}{3} \right\rceil$ for every graph G, which is best possible for the class of connected graphs in view of a disjoint union of cycles of length 3 to which bridges are added. For 2-connected graphs, though, this bound can be improved as follows.

Theorem 3. *If G is a 2-connected graph of order n, then* $\mathrm{lct}(G) \leqslant \left\lceil \frac{n}{3} - \frac{n^{2/3}}{36} \right\rceil$.

Proof. Let G be a 2-connected graph of order n. Let $\epsilon = \frac{1}{36}n^{-\frac{1}{3}}$ and $\tau = \left\lceil \left(\frac{1}{3} - \epsilon\right) n \right\rceil$. For a contradiction, we assume that $\mathrm{lct}(G) > \tau$. Let $C : u_0 \dots u_{\ell-1} u_0$ be a longest cycle of G. Since $V(C)$ as well as every set of $n - \ell + 1$ vertices of G are longest cycle transversals, we obtain

$$\left(\frac{1}{3} - \epsilon\right) n \leqslant \tau < \ell < n - \tau + 1 \leqslant \left(\frac{2}{3} + \epsilon\right) n + 1. \tag{3}$$

Since the set $T = \{u_0, \dots, u_{\tau-1}\}$ is too small to be a longest cycle transversal of G, there is a cycle $C' : v_0 \dots v_{\ell-1} v_0$ in $G - T$. Since G is 2-connected, the cycles C and C' intersect in at least two vertices. We may assume that $v_0 = u_r$ is the first and $v_k = u_s$ is the last common vertex of C and C' following the path $C - T$ from u_τ to $u_{\ell-1}$, that is $r < s$.

Since $v_0 \dots v_k u_{s+1} \dots u_{\ell-1} u_0 \dots u_{r-1}$ is a cycle of length at least $k+1+\tau$, we obtain $\ell-k-1 \geqslant \tau$. Since $v_{k+1} \dots v_{\ell-1} u_r \dots u_0 u_{\ell-1} \dots u_s$ is a cycle of length at least $\ell - (k-1) + \tau$, we obtain $k - 1 \geqslant \tau$.

Choosing three vertex-disjoint paths $P : x_1 \dots x_\tau$, $Q : y_1 \dots y_\tau$, and $R : z_1 \dots z_\tau$ as subpaths of the three paths $C[T]$, $C'[\{v_1, \dots, v_{k-1}\}]$, and $C'[\{v_{k+1}, \dots, v_{\ell-1}\}]$, respectively, we obtain the existence of two vertex-disjoint sets X and Y in $V(G) \setminus (V(P) \cup V(Q) \cup V(R))$ with $|X \cup Y| \leqslant n - 3\tau \leqslant 3\epsilon n$ such that X contains a path between some neighbors of every two of the vertices x_1, y_1, and z_1, and Y contains a path between some neighbors of every two of the vertices x_τ, y_τ, and z_τ. See Figure 3 for an illustration.

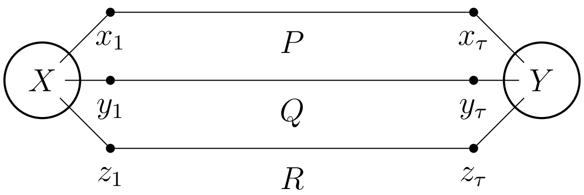

Figure 3: The paths P, Q, and R and the two sets X and Y.

If x_iy_j is an edge of G with $j \geqslant i$, then the path $y_1 \dots y_j x_i \dots x_\tau$, a path in Y between neighbors of x_τ and z_τ the path R, and a path in X between neighbors of y_1 and z_1 form a cycle of length at least $2\tau + (j - i + 1) + 2$. By (3), this implies that $j - i \leqslant \lceil 3\epsilon n \rceil - 3$. Using Lemma 1 as in the proof of Theorem 2, we infer that for every two distinct $A, B \in \{P, Q, R\}$, the bipartite graph $G_{A,B}$ with bipartition $V(A)$ and $V(B)$ formed by the edges between these two sets has a vertex cover $T_{A,B}$ of order less than $(3\epsilon n - 1)3\sqrt{2(\lceil 3\epsilon n \rceil - 3)}$ if $3\epsilon n > 2$ and of order 0 otherwise. Since every component of $G - (X \cup Y \cup T_{P,Q} \cup T_{P,R} \cup T_{Q,R})$ has order at most τ, the set $T' = X \cup Y \cup T_{P,Q} \cup T_{P,R} \cup T_{Q,R}$ is a longest cycle transversal of G of order less than $3\epsilon n + 3(3\epsilon n - 1)3\sqrt{2(\lceil 3\epsilon n \rceil - 3)}$ if $3\epsilon n > 2$ and of order at most $3\epsilon n$ otherwise. For $\epsilon = \frac{1}{36} n^{-\frac{1}{3}}$, it follows that $|T'| \leqslant \left(\frac{1}{3} - \epsilon\right) n$, which yields the final contradiction. $\square$

Since every two longest paths of a connected graph G intersect, it follows that $\mathrm{lpt}(G) \leqslant \left\lceil \frac{|\mathcal{P}(G)|}{2} \right\rceil$. Similarly, if every k longest paths of a connected graph G would intersect for some $k \geqslant 3$, then it would follow that $\mathrm{lpt}(G) \leqslant \left\lceil \frac{|\mathcal{P}(G)|}{k} \right\rceil$. The next result shows how to decrease the multiplicative constant $1/2$ in the former bound at the cost of adding a square-root proportion of the total number of vertices in the graph.

Proposition 4. *If G is a connected graph and $\alpha \geqslant 2$, then*

$$\mathrm{lpt}(G) \leqslant \frac{|\mathcal{P}(G)|}{\alpha} + \sqrt{\alpha \, |V(G)|}.$$

Proof. We proceed by induction on the order n of G, the statement being true if $n = 1$. Let $n \geqslant 2$ and assume that the statement holds for all connected graphs of order less than n. Let G be a connected graph of order n and let ℓ be the order of the longest paths in G.

We may assume that $|\mathcal{P}(G)| > \sqrt{\alpha n}$ since otherwise we obtain a longest path transversal of the desired size by picking one vertex in each longest path of G. Next, since the vertex set of a longest path in G is a longest path transversal, we may also assume that $\ell > \sqrt{\alpha n}$.

For a vertex $v \in V(G)$, let p_v be the number of paths in $\mathcal{P}(G)$ that contain v. We may assume that $p_v < \alpha$ for every vertex $v \in V(G)$. Indeed, suppose that v is a vertex such that $p_v \geqslant \alpha$. In particular, $p_v \geqslant 1$. If the set $\{v\}$ is a longest path transversal of G, then G satisfies the desired property. Otherwise let $G' = G - v$ and note that G' contains a path of order ℓ. Furthermore, $|\mathcal{P}(G')| = |\mathcal{P}(G)| - p_v \leqslant |\mathcal{P}(G)| - \alpha$. Note that all paths of order ℓ in G' must belong to the same component of G', since every two longest paths intersect. Let C be this component; thus $\mathcal{P}(C) = \mathcal{P}(G')$. The induction hypothesis applied to C yields that $\mathrm{lpt}(G') \leqslant \frac{|\mathcal{P}(G)|}{\alpha} - 1 + \sqrt{\alpha(n-1)}$. As $\mathrm{lpt}(G) \leqslant \mathrm{lpt}(G') + 1$, we deduce that $\mathrm{lpt}(G) \leqslant \frac{|\mathcal{P}(G)|}{\alpha} + \sqrt{\alpha n}$.

We now consider the number N of pairs (v, P) such that $P \in \mathcal{P}(G)$ and $v \in V(P)$. One the one hand, since $N = \sum_{v \in V(G)} p_v$, we deduce from the previous observations that $N < \alpha n$. On the other hand, since $N = \sum_{P \in \mathcal{P}(G)} |V(P)| = \ell\,|\mathcal{P}(G)|$, the previous observations also imply that $N > \alpha n$. This contradiction concludes the proof. $\square$

The minimum sizes of transversals of longest paths can be bounded in classes of graphs with small separators, such as planar graphs and graphs of bounded tree-width. As before, no effort is made to minimize the constant multiplicative factors appearing in the next two results.

Proposition 5. *If G is a connected planar graph of order at least 2, then*

$$\operatorname{lpt}(G) \leqslant 9\sqrt{|V(G)|}\log|V(G)|\,.$$

Proof. We proceed by induction on the order n of G, the result being true if $n = 2$. Let $n \geqslant 3$ and assume that the statement holds for all connected planar graphs of order at least 2 and less than n. Let G be a connected planar graph of order n and let ℓ be the order of the longest paths in G. In particular, $\ell \geqslant 2$. Since G is planar, the separator theorem of Lipton and Tarjan ensures that G contains a set X of order at most $2\sqrt{2}\sqrt{n}$ such that every component of $G - X$ has order at most $2n/3$.

If X is a longest path transversal of G, then G satisfies the desired property. Otherwise, $G - X$ contains a path of order ℓ. Note that, since every two longest paths of G intersect, all paths of order ℓ in $G - X$ must be contained in the same component of $G - X$, which we call C. Moreover, the order of C is at most $2n/3$ and at least ℓ, so the induction hypothesis implies that C has a longest path transversal X' of order at most $9\sqrt{2n/3}\log(2n/3)$. Therefore, $X \cup X'$ is a longest path transversal of G of order at most

$$\begin{aligned} 2\sqrt{2}\sqrt{n} + 9\sqrt{2n/3}\log(2n/3) &= 9\sqrt{2n/3}\log n + \sqrt{n}\cdot\left(2\sqrt{2} - 9\sqrt{2/3}\log(3/2)\right) \\ &\leqslant 9\sqrt{n}\log n \end{aligned}$$

since $9\sqrt{2/3}\log(3/2) > 2\sqrt{2}$. This concludes the proof. $\square$

An anlaguous statement is true for graphs of bounded tree-width. Indeed, if G is a graph with tree-width at most k, then there is a set X of vertices of G of order at most $k+1$ such that every component of $G - X$ has order at most $|V(G)|/2$. Consequently, an inductive reasoning similar to that made in the proof of Proposition 5 yields the following statement.

Proposition 6. *If G is a connected graph of tree-width at most k and order at least 2, then*

$$\operatorname{lpt}(G) \leqslant 3k\log|V(G)|\,.$$

We proceed to circular arc graphs. We explain the approach of Balister *et al.* [1], the problem with their argument, and prove the following weaker result.

Theorem 7. *Let G be a circular-arc graph.*

If G is connected, then $\operatorname{lpt}(G) \leqslant 3$, and if G is 2-connected, then $\operatorname{lct}(G) \leqslant 3$.

Let G be a connected circular arc graph. Let C be a circle and let $\mathcal{F}$ be a collection of open arcs of C such that G is the intersection graph of $\mathcal{F}$. In view of the result for interval graphs mentioned in the introduction, we may assume that $C \subseteq \bigcup_{A\in\mathcal{F}} A$. Furthermore, we may assume that all endpoints of arcs in $\mathcal{F}$ are distinct.

Balister *et al.* [1] consider a collection $\mathcal{K} = \{K_0, \ldots, K_{n-1}\}$ of arcs in $\mathcal{F}$ such that

(0) $C \subseteq \bigcup_{A\in\mathcal{K}} A$,

(1) n is minimal, and

(2) each K_i is maximal, that is, no arc in $\mathcal{F}$ properly contains an arc in $\mathcal{K}$.

They may assume that $n \geqslant 2$, because otherwise, G has a universal vertex that belongs to every longest path or cycle. We consider the indices of the arcs in $\mathcal{K}$ as elements of $\mathbf{Z}_n$, that is, modulo n.

A *chain of order ℓ in $\mathcal{F}$* is a sequence $\mathcal{P} : A_1 \ldots A_\ell$ of distinct arcs in $\mathcal{F}$ such that $A_i \cap A_{i+1} \neq \emptyset$ for $i \in [\ell - 1]$. The chain $\mathcal{P}$ is *closed*, if $A_\ell \cap A_1 \neq \emptyset$. Thus chains and closed chains in $\mathcal{F}$ correspond to paths and cycles in G. For a chain $\mathcal{P} : A_1 \ldots A_\ell$ in $\mathcal{F}$, let $\mathcal{K}(\mathcal{P}) = \{A_1, \ldots, A_\ell\} \cap \mathcal{K}$.

If $\mathcal{P} : A_1 \ldots A_\ell$ is a chain in $\mathcal{F}$ of largest order, then Balister *et al.* [1, Lemma 3.1] proved that $\mathcal{K}(\mathcal{P})$ is of the form $\{K_i : i \in I\}$ for some contiguous and non-empty subset I of $\mathbf{Z}_n$. Their argument actually yields the same statement for closed chains, that is, if $\mathcal{C}$ is a closed chain in $\mathcal{F}$ of largest order, then $\mathcal{K}(\mathcal{C})$ is of the form $\{K_i : i \in J\}$ for some contiguous and non-empty subset J of $\mathbf{Z}_n$.

In the proof of their main result [1, Theorem 3.3] — stating that $\mathrm{lpt}(G) = 1$ — Balister *et al.* choose a chain $\mathcal{P}$ in $\mathcal{F}$ of largest order such that $\mathcal{K}(\mathcal{P})$ has minimum order. They let $\mathcal{K}(\mathcal{P})$ be $\{K_{a+1}, \ldots, K_{b-1}\}$ and assert that K_{b-1} belongs to $\mathcal{K}(\mathcal{Q})$ for every chain $\mathcal{Q}$ in $\mathcal{F}$ of largest order, that is, the vertex of G corresponding to the arc K_{b-1} would belong to every longest path of G.

For a contradiction, they consider a chain $\mathcal{Q}$ in $\mathcal{F}$ of largest order such that $K_{b-1} \notin \mathcal{K}(\mathcal{Q})$. They set $\mathcal{K}(\mathcal{Q}) = \{K_{\ell+1}, \ldots, K_{m-1}\}$. They deduce from the choice of $\mathcal{P}$ that $K_{\ell+1} \in \mathcal{K}(\mathcal{Q}) \setminus \mathcal{K}(\mathcal{P})$ since $K_{b-1} \in \mathcal{K}(\mathcal{P}) \setminus \mathcal{K}(\mathcal{Q})$. Using their Lemma 3.2 [1], they reorder the arcs in the chains $\mathcal{P}$ and $\mathcal{Q}$ and obtain chains $\mathcal{P}^*$ and $\mathcal{Q}^*$ containing the same arcs as $\mathcal{P}$ and $\mathcal{Q}$ in a possibly different order, respectively. They split these chains at K_{b-1} and $K_{\ell+1}$ writing them as $\mathcal{P}^* : \mathcal{P}_1 K_{b-1} \mathcal{P}_2$ and $\mathcal{Q}^* : \mathcal{Q}_1 K_{\ell+1} \mathcal{P}_2$, respectively.

Their core statement is that $\mathcal{C}_1 : \mathcal{P}_1 K_{b-1} \mathcal{R} K_{\ell+1} \mathcal{Q}_1^r$ and $\mathcal{C}_2 : \mathcal{P}_2^r K_{b-1} \mathcal{R} K_{\ell+1} \mathcal{Q}_2$ are chains that satisfy the inequality $|\mathcal{C}_1| + |\mathcal{C}_2| \geqslant 2 + |\mathcal{P}| + |\mathcal{Q}|$, where $\mathcal{R}$ is the possibly empty chain $K_b \ldots K_\ell$ and the exponent "r" means reversal. In order to prove this statement, they have to show that no arc appears twice in these sequences. They give details only for $\mathcal{C}_1$. Their argument that $\mathcal{C}_1$ is a chain heavily relies on the properties of the reordered chains $\mathcal{P}^*$ and $\mathcal{Q}^*$ guaranteed by their Lemma 3.2. In the proof of Lemma 3.2 these properties are established by iteratively shifting within $\mathcal{P}$ the arc K_{b-1} to the beginning of $\mathcal{P}$ and, similarly, by iteratively shifting within $\mathcal{Q}$ the arc $K_{\ell+1}$ to the beginning of $\mathcal{Q}$. After proving that $\mathcal{C}_1$ is indeed a chain, they say that the same type of argument shows that $\mathcal{C}_2$ is a chain as well.

This is the gap in their argument.

In order to use the same type of argument for $\mathcal{C}_2$, they would need reversed versions of the properties guaranteed by Lemma 3.2, that is, in order to establish these properties they would have to iteratively shift within $\mathcal{P}$ the arc K_{b-1} to the end of $\mathcal{P}$ and, similarly, to iteratively shift within $\mathcal{Q}$ the arc $K_{\ell+1}$ to the end of $\mathcal{Q}$. This may easily result in reorderings that are distinct from $\mathcal{P}^*$ and $\mathcal{Q}^*$. In view of this asymmetry, the suitably adapted chain $\mathcal{C}_2$, which would use the different reorderings of $\mathcal{P}$ and $\mathcal{Q}$, need not satisfy the crucial inequality $|\mathcal{C}_1|+|\mathcal{C}_2| \geqslant 2+|\mathcal{P}|+|\mathcal{Q}|$ and the argument breaks down.

We proceed to the proof of our Theorem 7.

Proof of Theorem 7. Let G be a connected circular arc graph. We choose C, $\mathcal{F}$, and $\mathcal{K}$ exactly as above and we start by proving the following statement.

Assertion. *If $\mathcal{P}$ and $\mathcal{Q}$ are chains of largest order in $\mathcal{F}$ such that*

$$\mathcal{K}(\mathcal{P}) = \{K_{a+1}, \dots, K_{b-1}\} = \{K_i : i \in I(\mathcal{P})\} \quad \textit{and}$$
$$\mathcal{K}(\mathcal{Q}) = \{K_{\ell+1}, \dots, K_{m-1}\} = \{K_i : i \in I(\mathcal{Q})\}$$

are disjoint, then $a = m-1$ or $b = \ell+1$, that is, the subsets $I(\mathcal{P})$ and $I(\mathcal{Q})$ of $\mathcal{Z}_n$ are contiguous.

To establish this assertion, assume on the contrary that $a \neq m-1$ and $b \neq \ell+1$. Select a set $S(\mathcal{P})$ of points of C such that $S(\mathcal{P})$ contains a point in the intersection of every two consecutive arcs of $\mathcal{P}$. Define $S(\mathcal{Q})$ analogously. If K_a or K_b would intersect $S(\mathcal{P})$ or $S(\mathcal{Q})$, then K_a or K_b could be inserted into $\mathcal{P}$ or $\mathcal{Q}$, respectively, contradicting the assumption that these chains are of largest order. If $S(\mathcal{P})$ or $S(\mathcal{Q})$ would intersect both arcs of $C \setminus (K_a \cup K_b)$, then some arc of $\mathcal{P}$ or $\mathcal{Q}$ would properly contain K_a or K_b, which yields a contradiction to the condition (2) in the choice of $\mathcal{K}$. Since $\mathcal{K}(\mathcal{P})$ and $\mathcal{K}(\mathcal{Q})$ are disjoint, the sets $S(\mathcal{P})$ and $S(\mathcal{Q})$ are contained in different of the two arcs of $C \setminus (K_a \cup K_b)$. Since G is connected, $\mathcal{P}$ and $\mathcal{Q}$ have a common arc A. This arc A intersects $S(\mathcal{P})$ as well as $S(\mathcal{Q})$, that is, it intersects both arcs of $C \setminus (K_a \cup K_b)$. Hence either K_a or K_b is properly contained in A, which again yields a contradiction to the condition (2) in the choice of $\mathcal{K}$. This conludes the proof of the assertion.

Again let $\mathcal{P}$ be a chain in $\mathcal{F}$ of largest order and, subject to this, such that $\mathcal{K}(\mathcal{P})$ has minimum order. Let $\mathcal{K}(\mathcal{P}) = \{K_{a+1}, \dots, K_{b-1}\}$.

In view of the desired statement, we may assume that $\mathcal{F}$ contains a chain $\mathcal{Q}$ of largest order such that $K_{a+1}, K_{b-1} \notin \mathcal{K}(\mathcal{Q})$. Among all such chains, we assume that $\mathcal{Q}$ is chosen such that $\mathcal{K}(\mathcal{Q})$ has minimum order. Let $\mathcal{K}(\mathcal{Q}) = \{K_{\ell+1}, \dots, K_{m-1}\}$. By the choice of $\mathcal{P}$, the sets $\mathcal{K}(\mathcal{P})$ and $\mathcal{K}(\mathcal{Q})$ are disjoint. By the assertion, we may assume that $b = \ell+1$.

In view of the desired statement, we may assume that $\mathcal{F}$ contains a chain $\mathcal{R}$ of largest order such that $K_{a+1}, K_{b-1}, K_{m-1} \notin \mathcal{K}(\mathcal{R})$. Among all such chains, we assume that $\mathcal{R}$ is chosen such that $\mathcal{K}(\mathcal{R})$ has minimum order. Let $\mathcal{K}(\mathcal{R}) = \{K_{p+1}, \dots, K_{q-1}\}$. By the choice of $\mathcal{P}$ and $\mathcal{Q}$, the sets $\mathcal{K}(\mathcal{P}) \cup \mathcal{K}(\mathcal{Q})$ and $\mathcal{K}(\mathcal{R})$ are disjoint. Applying the assertion to $\mathcal{P}$ and $\mathcal{R}$ as well as to $\mathcal{Q}$ and $\mathcal{R}$, we obtain $p = m-1$ and $q = a+1$, that is, $\mathcal{K}(\mathcal{P}) \cup \mathcal{K}(\mathcal{Q}) \cup \mathcal{K}(\mathcal{R})$ is a partition of $\mathcal{K}$.

In view of the desired statement, we may assume that $\mathcal{F}$ contains a chain $\mathcal{S}$ of largest order such that $K_{a+1}, K_{\ell+1}, K_{p+1} \notin \mathcal{K}(\mathcal{S})$. We deduce from the choice of $\mathcal{P}$ that $\mathcal{K}(\mathcal{S})$ has at least as many elements as $\mathcal{K}(\mathcal{P})$. This implies that $\mathcal{K}(\mathcal{S})$ is disjoint from $\mathcal{K}(\mathcal{P})$. Now, by the choice of $\mathcal{Q}$, this implies that $\mathcal{K}(\mathcal{S})$ has at least as many elements as $\mathcal{K}(\mathcal{Q})$. This in turn implies that the set $\mathcal{K}(\mathcal{S})$ is disjoint from $\mathcal{K}(\mathcal{P}) \cup \mathcal{K}(\mathcal{Q})$. Finally, by the choice of $\mathcal{R}$, this implies that $\mathcal{K}(\mathcal{S})$ has at least as many elements as $\mathcal{K}(\mathcal{R})$. This in turn implies that the set $\mathcal{K}(\mathcal{S})$ equals $\mathcal{K}(\mathcal{R})$, that is, $\mathcal{K}(\mathcal{S})$ contains K_{p+1}, which is a contradiction.

This completes the proof that $\mathrm{lpt}(G)$ is at most 3.

From now on we assume that G is a 2-connected circular arc graph, that is, every two longest cycles in G — closed chains of largest order in $\mathcal{F}$ — intersect. It is straightforward to see that the assertion also applies to closed chains instead of chains. Arguing exactly as above for closed chains in $\mathcal{F}$ instead of chains in $\mathcal{F}$ implies that $\mathrm{lct}(G) \leqslant 3$. $\square$

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